Computation of the first Stiefel-Whitney class of $\overline{\mathcal{M}_{0,n}^{\mathbb{R}}}$

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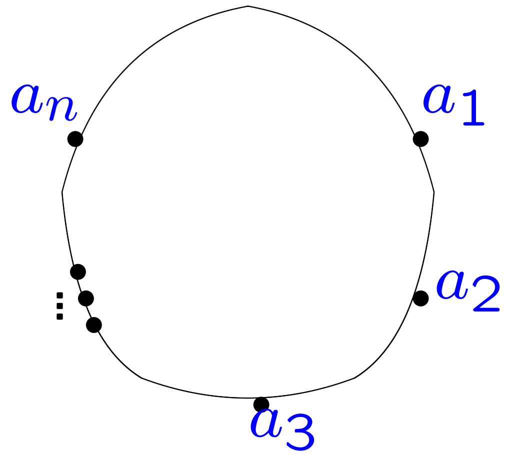
This is a joint work with Natalia Amburg.

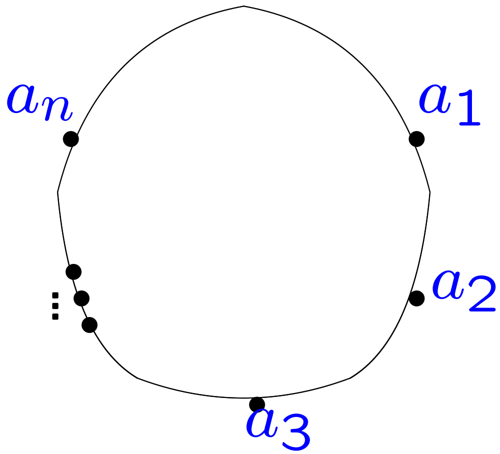
$\mathcal{M}_{0,n}^{\mathbb{R}} = \Big\{$



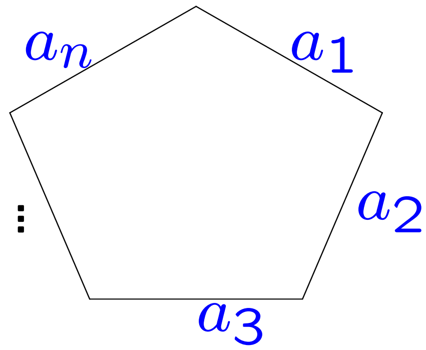
$\Big\} / \sim$

— real algebraic curves of genus 0 with n marked and numbered points.

$$\mathcal{M}^{\mathbb{R}}_{0,n} = \Bigg\{$$



$$=$$



$$\Bigg\} / \sim$$

— real algebraic curves of genus 0 with n marked and numbered points.

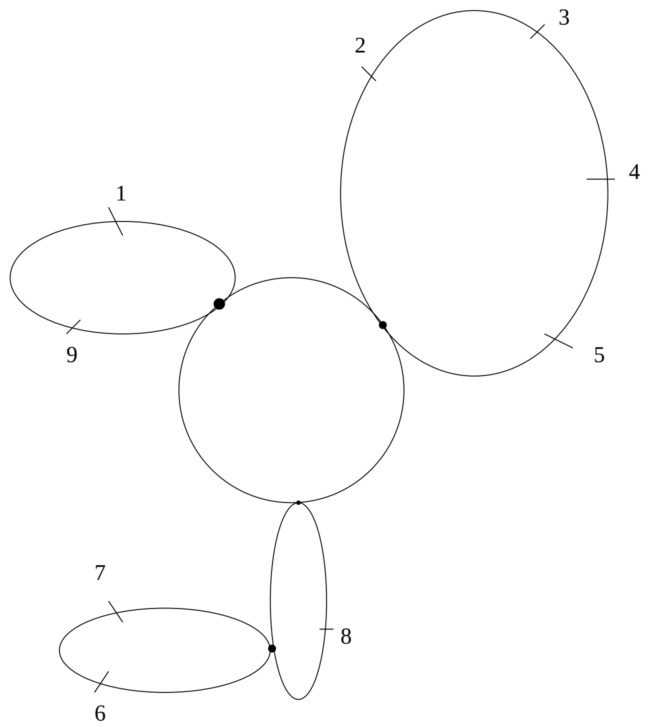
$\overline{\mathcal{M}_{0,n}^{\mathbb{R}}}$ is the *Deligne-Mumford compactification* of $\mathcal{M}_{0,n}^{\mathbb{R}}$

I.e., moduli of "cacti-like" structures: 3-dimensional "trees" of flat circles with the points $\{1, 2, \ldots, n\}$ on them.

Definition. A *stable curve* of genus 0 with n marked points over $\mathbb{R}$ is $C = C_1 \cup C_2 \cup \ldots \cup C_p$ with n different marked points $z_1, z_2, \ldots, z_n \in C$, s.t.

- $\forall$ z_i $\exists$! line C_j : $z_i \in C_j$.
- $\forall$ pair C_i, C_j, $C_i \cap C_j$ is either $\emptyset$ or $\{X\}$, and it is transversal.
- The graph of C ($C_1, C_2, \ldots, C_p$ are vertices; edges are intersections) is a tree.
- The number of special points (marked or intersection) in C_j ≥ 3 $\forall$ $j = 1, \ldots, p$.

p is the *number of components*.

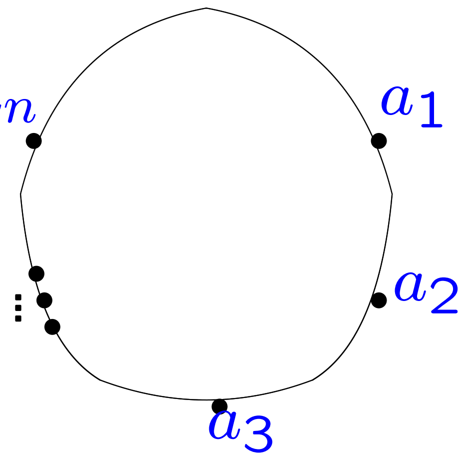


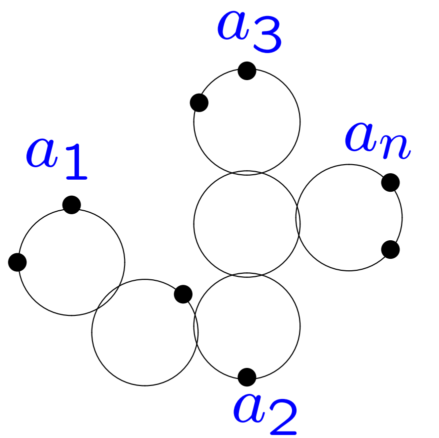
A stable curve over $\mathbb{R}$ of genus 0 with 9 marked points

Definition.

Let $C = (C_1, C_2, \ldots, C_p, z_1, z_2, \ldots, z_n)$ and $C' = (C'_1, C'_2, \ldots, C'_p, z'_1, z'_2, \ldots, z'_n)$ be stable curves of genus 0 with n points. C, C' are called *equivalent* if $\exists$ iso. $f : C \to C'$: $f(z_i) = z'_i$ $\forall\ i = 1, \ldots, n$.

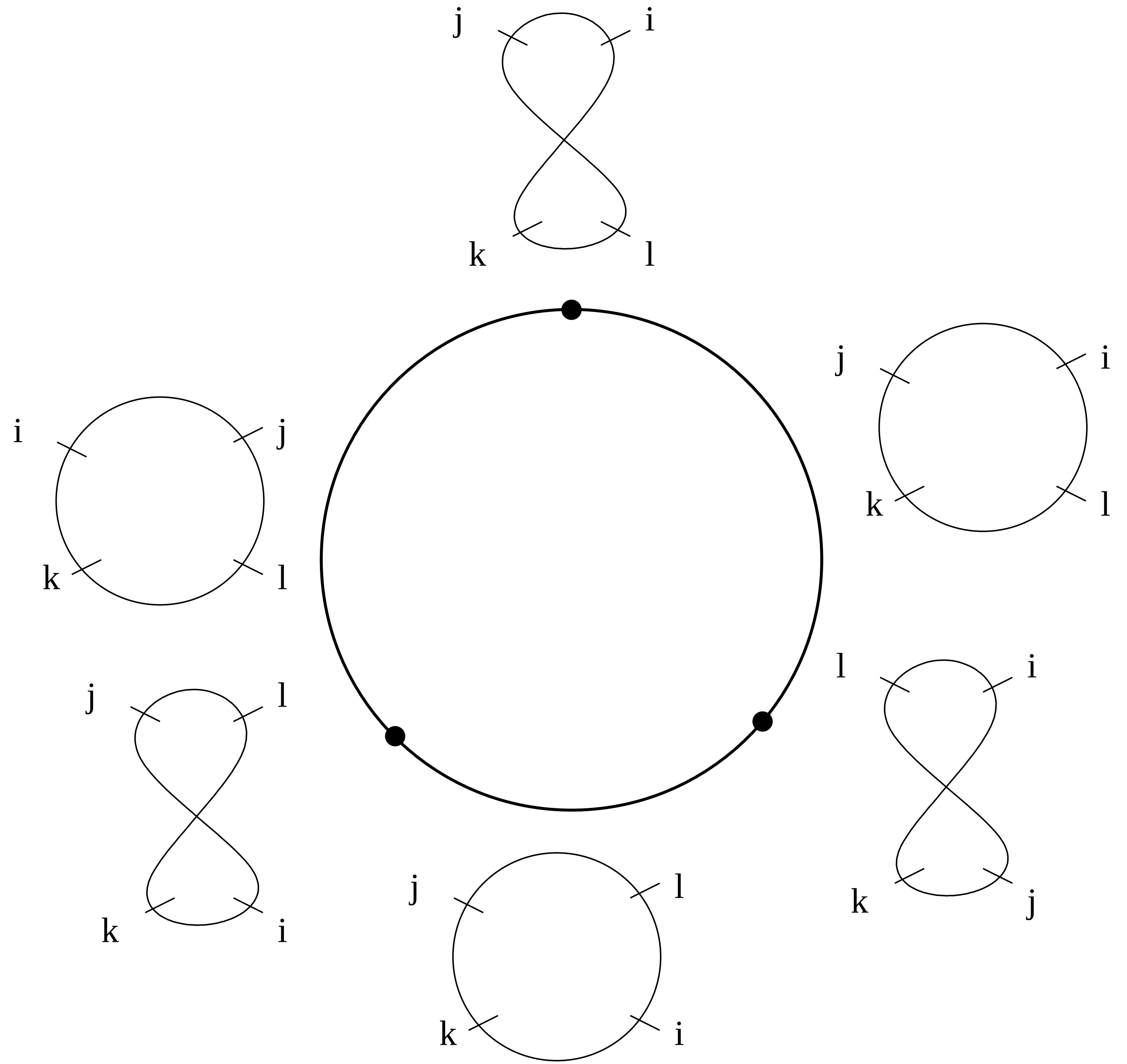
Definition. $n \geq 3$. *Deligne-Mumford compactification* of

$\Bigg\{$ 

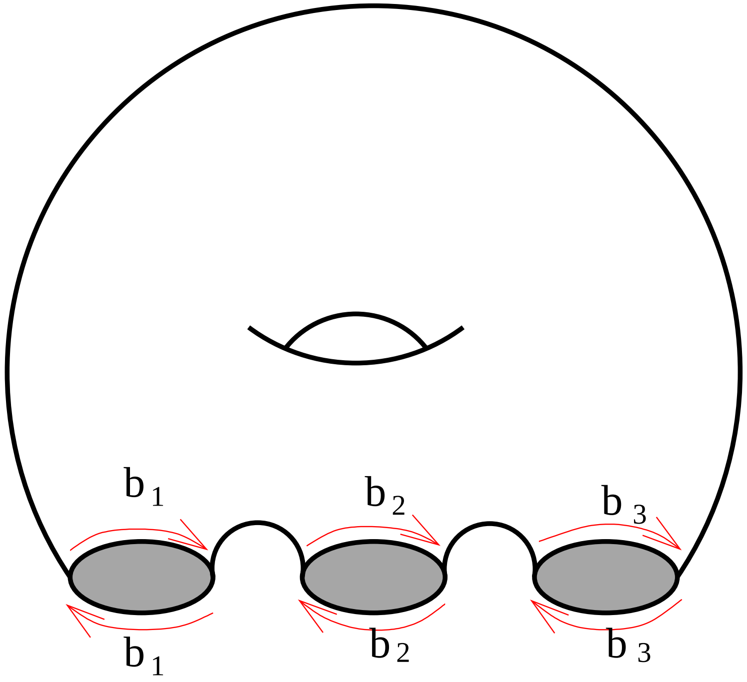
$\Bigg\}/\sim$ is $\Bigg\{$ 

$\Bigg\}/\sim$

Theorem. [S. Devadoss, 1999] Let $n > 4$. Then the space $\overline{\mathcal{M}^{\mathbb{R}}_{0,n}}$ is a non-orientable compact variety of real dimension $\dim(\overline{\mathcal{M}^{\mathbb{R}}_{0,n}}) = n - 3$.



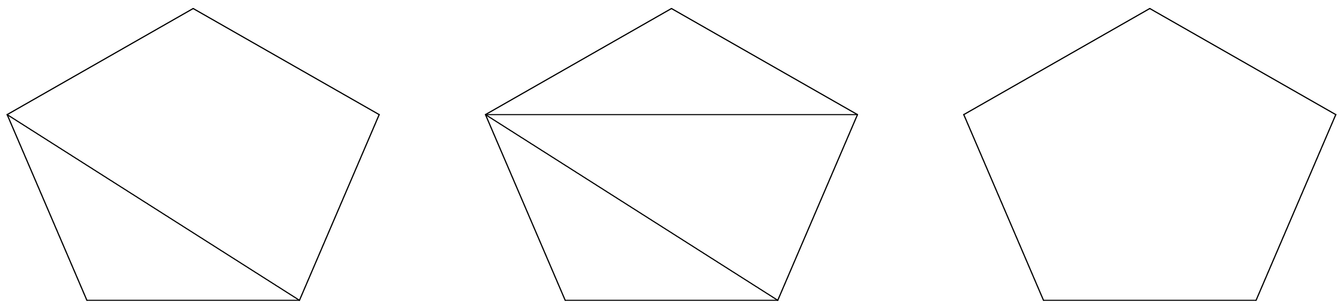
$\overline{\mathcal{M}}^{\mathbb{R}}_{0,4}$



$\overline{\mathcal{M}_{0,5}^{\mathbb{R}}}$

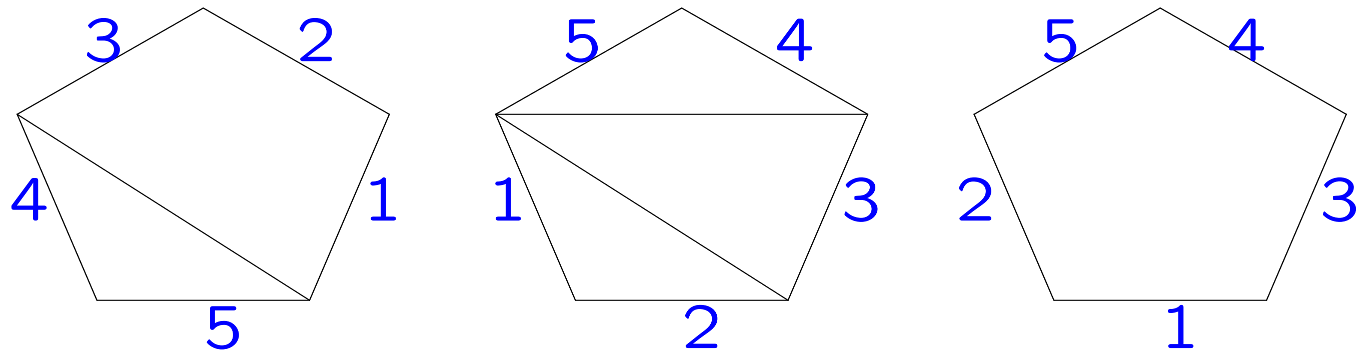
Cell decomposition of $\overline{\mathcal{M}_{0,n}^{\mathbb{R}}}$ [S. Devadoss, 1999]

1. Consider right n-gons, possibly, with several non-intersecting outside the vertices diagonals:



1. Consider right n-gons, possibly, with several non-intersecting outside the vertices diagonals:

2. Mark their edges by $1, 2, \ldots, n$



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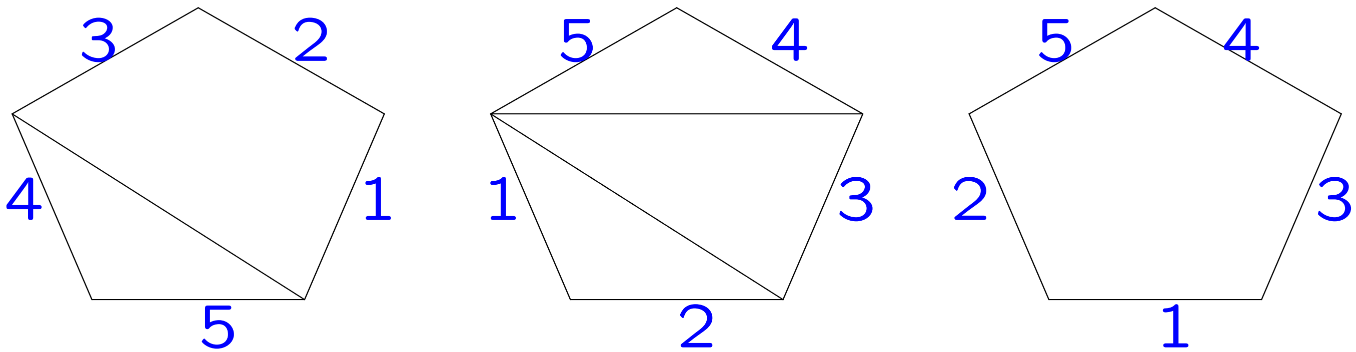
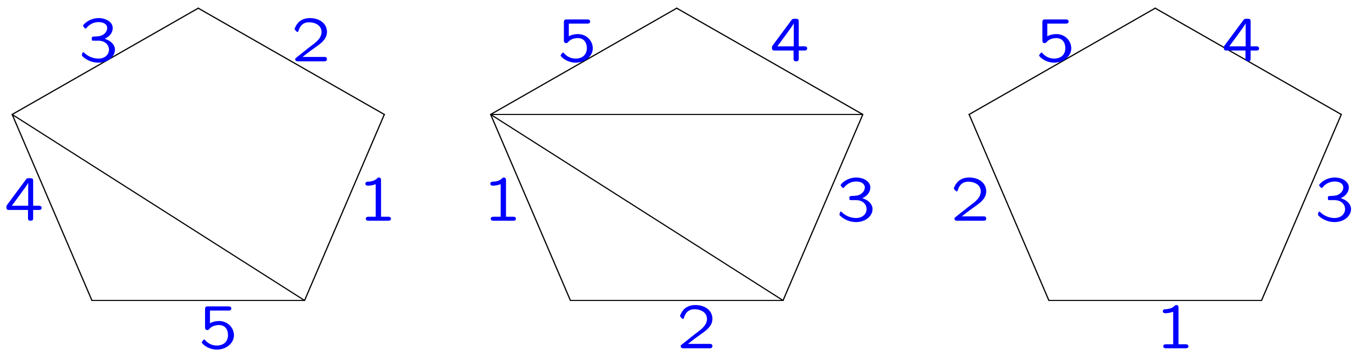


3. Identify polygons that be transformed to each other by the dihedral group action.

1. Right n-gons

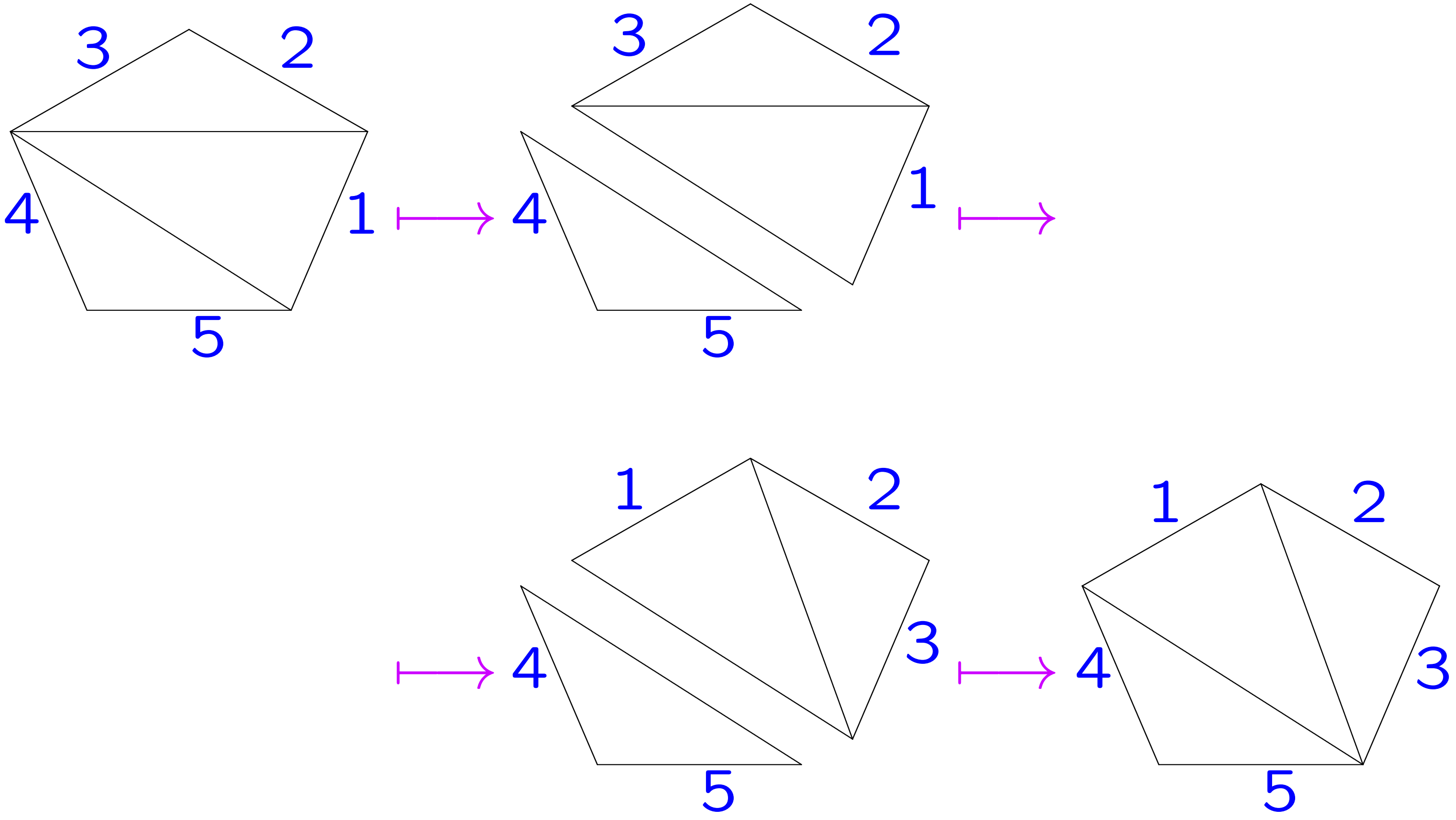
2. Mark their edges by $1, 2, \ldots, n$



3. Action of the dihedral group — the same.

4. Identify polydons that be transformed to each other by the series of twist operations.

Twist operation:



Cells of $\overline{\mathcal{M}_{0,n}^{\mathbb{R}}}$ — such n-gons

Cells of $\overline{\mathcal{M}^{\mathbb{R}}_{0,n}}$ — such n-gons

Marked points — sides of n-gons

Intersection points — diagonals

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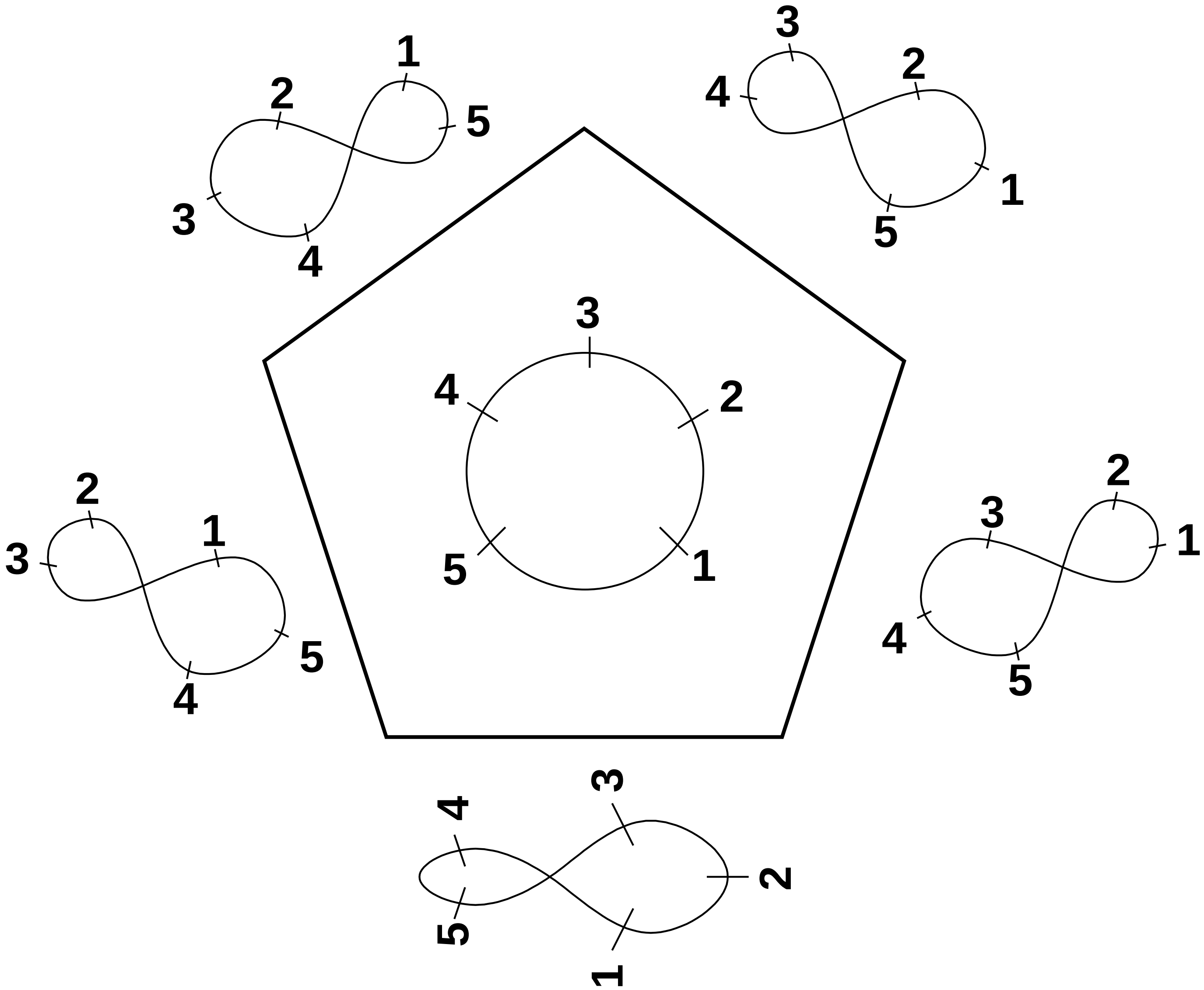
cells of codim 2 — n-gons with 2 diagonal — 3-component stable curves

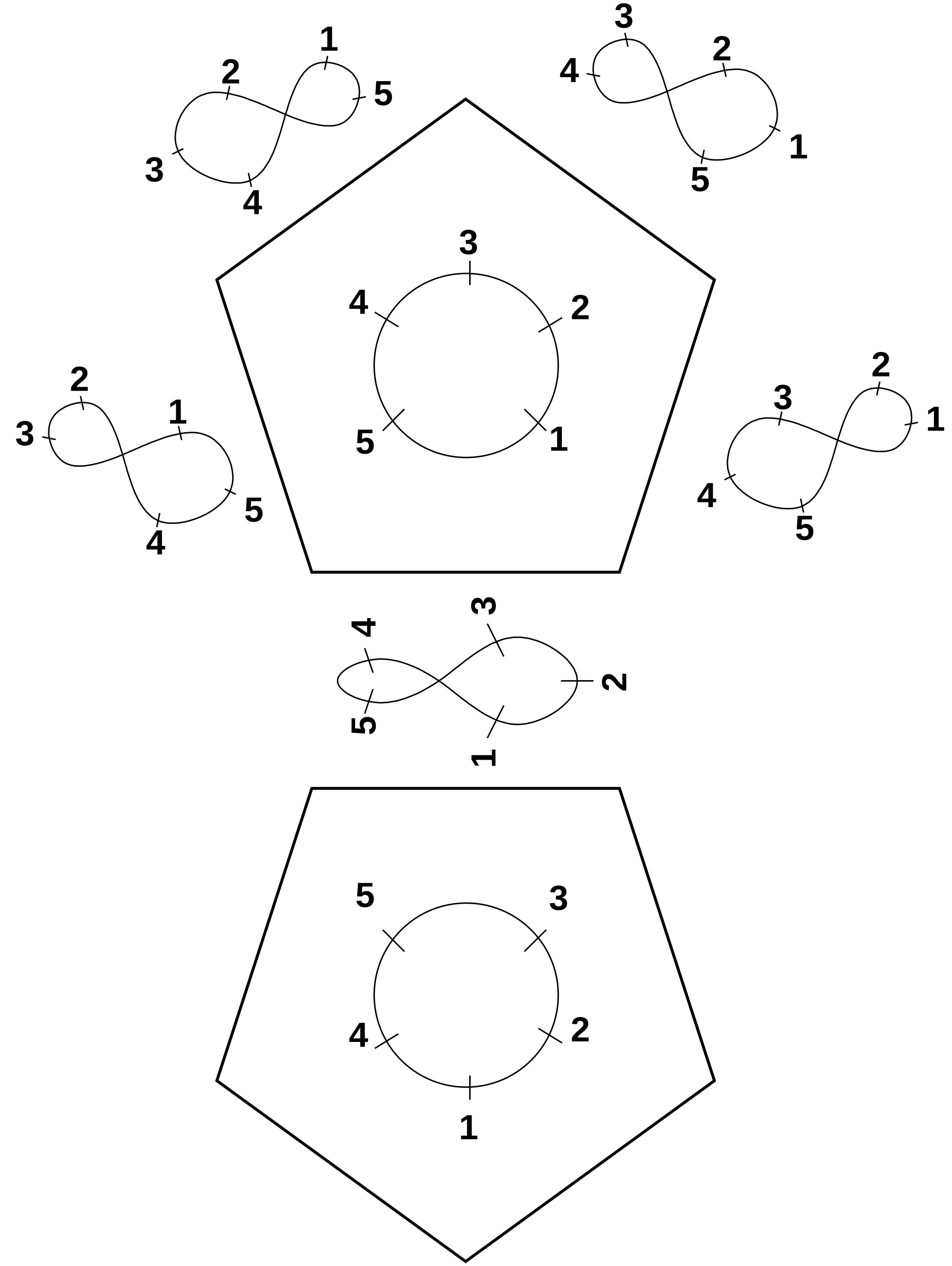
cells of codim k — n-gons with k diagonal — $k+1$-component stable curves

- The graph of C is a tree — diagonals do not intersect inside the polygon

- The number of special points in $C_j \geq 3 \; \forall \; j = 1, \ldots, p$ — diagonals are diagonals, i.e., each part of the "big" polygon is at least a 3-gon

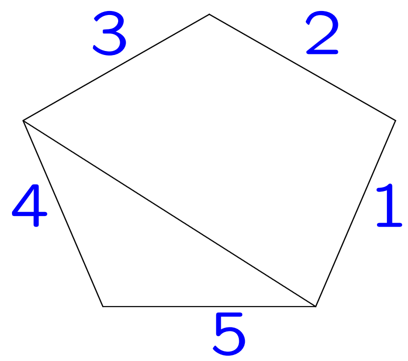


One of cells of $\overline{\mathcal{M}^{\mathbb{R}}_{0,5}}$.

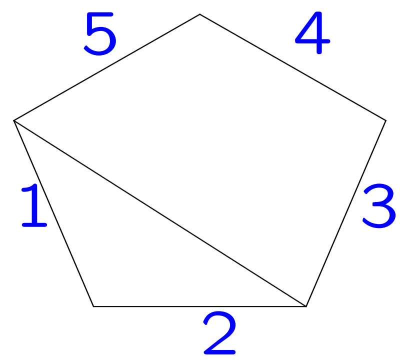


Two adjoint cells of $\overline{\mathcal{M}^{\mathbb{R}}_{0,5}}$.

Boundary cells:

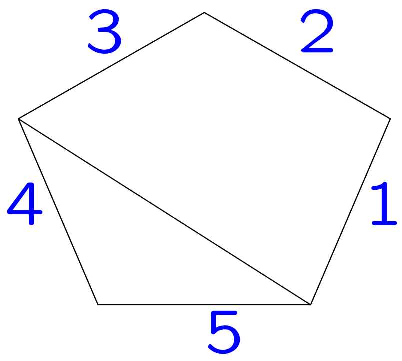


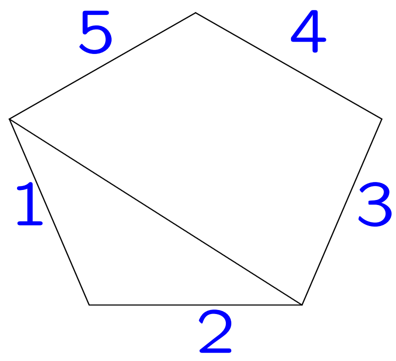
Bottom edge



Right-lower edge

Boundary cells:

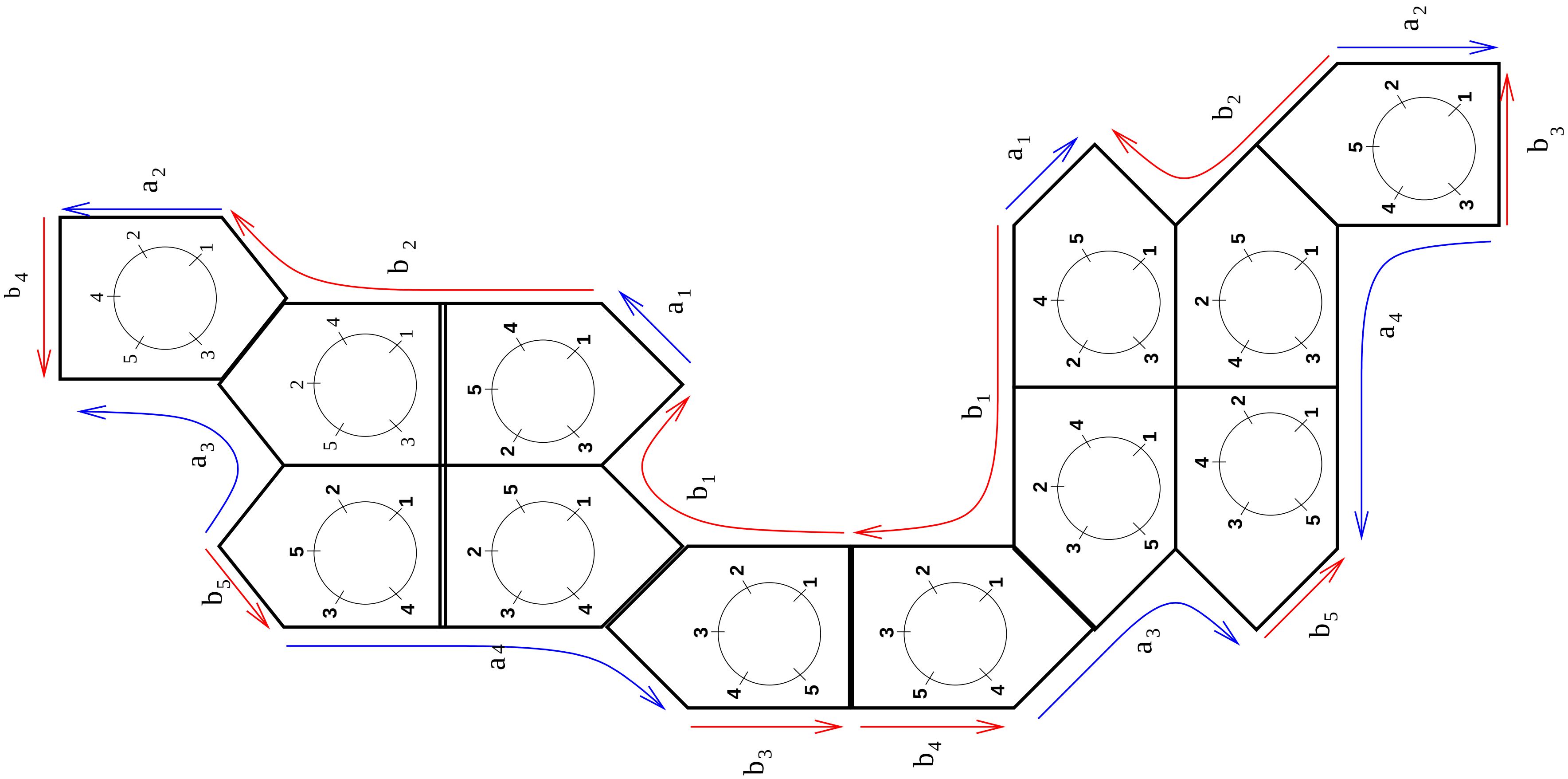




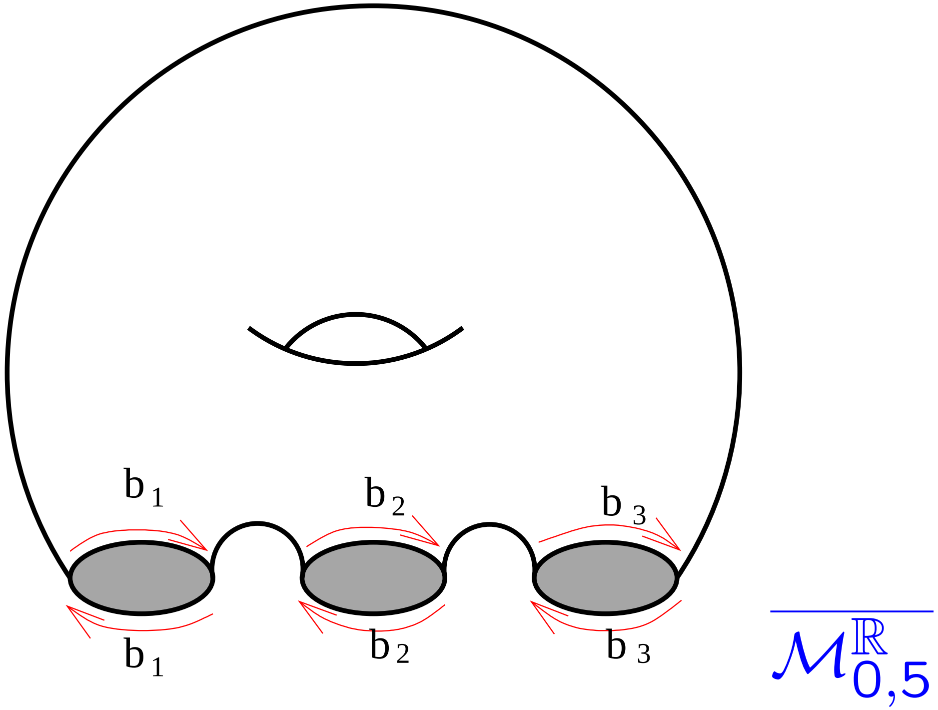
Bottom edge   Right-lower edge

Proposition. Cell decomposition of $\overline{\mathcal{M}_{0,n}^{\mathbb{R}}}$ contains $\frac{(n-1)!}{2}$ cells of the maximal dimension $n-3$.

$n = 5 \Longrightarrow 12$ cells



a2
b4
b2
a1
a3
b1
b5
a4
b3
b4
a3
a1
b1
b2
a2
b3
a4
b5



b 1
b 2
b 3
b 1
b 2
b 3
$\overline{\mathcal{M}_{0,5}^{\mathbb{R}}}$

Stiefel-Whitney class of $\overline{\mathcal{M}_{0,n}^{\mathbb{R}}}$

StWh are topological invariants of a real vector bundles that describe the obstructions to constructing everywhere independent sets of sections of the vector bundle, is a $\mathbb{Z}/2\mathbb{Z}$-characteristic class associated to real vector bundles.

StWh are indexed from 0 to d — the dimension of the vector space fiber of the vector bundle.

StWh$\neq 0$ for some $i \Rightarrow \nexists\ (n-i+1)$ everywhere linearly independent sections of the vector bundle.

$0 \neq n$'th StWh indicates that $\forall$ section of the bundle must vanish at some point.

$0 \neq 1$'st StWh indicates that the vector bundle is not orientable.

We consider the homological class W_{n-4}, which is Poincaré dual to the 1st StWh class of $\overline{\mathcal{M}^{\mathbb{R}}_{0,n}}$.

Theorem. [Milnor, Stasheff] Let M be a smooth compact variety without a boundary, K be a cell decomposition of M, $k_j \subset K$ denote the cells of the maximal dimension d. Let us fix the orientation on the cells of MAX dim $\overline{k_j}$. Then

$$W_{d-1}(M) = \left(\frac{1}{2}\sum \partial\overline{k_j}\right) \mod 2.$$

Theorem. [AK, 2014]

$n \geq 5$, $\mathcal{M}^{\mathbb{R}}_{0,n}$, points $\{1, 2, \ldots, n\}$.

$\overline{\mathcal{M}^{\mathbb{R}}_{0,n}}$ is Deligne-Mumford compactification.

Then $W_{n-4}(\overline{\mathcal{M}^{\mathbb{R}}_{0,n}})$ consists exactly from those cells of codim 1, that satisfy: irreducible component of the curve which contains ≤ 1 point from the set $\{1, 2, 3\}$ contains an odd number of points from $\{1, 2, \ldots, n\}$.

Corollary. $n \geq 6$ is even, $\overline{\mathcal{M}_{0,n}^{\mathbb{R}}}$ is the Deligne-Mumford compactification of $\mathcal{M}_{0,n}^{\mathbb{R}}$.

Then $W_{n-4}(\overline{\mathcal{M}_{0,n}^{\mathbb{R}}})$ consists exactly from the cells of codim 1, such that each irreducible component of the curve contains an odd number of marked points.

Definition. A *coordinate map* on the space $\mathcal{M}_{0,n}^{\mathbb{R}}$ is

$$\varphi : \mathcal{M}_{0,n}^{\mathbb{R}} \to \mathbb{R}^{n-3}$$

Let $(\mathbb{P}_1(\mathbb{R}), z_1, \ldots, z_n) \in \mathcal{M}_{0,n}^{\mathbb{R}}$, $z_i \in \mathbb{P}_1(\mathbb{R})$, we fix

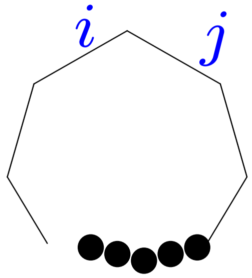
$$z_1 = 0, z_2 = 1, z_3 = \infty$$

Then

$$\varphi(\mathbb{P}_1(\mathbb{R}), z_1, \ldots, z_n) = (z_4, \ldots, z_n).$$

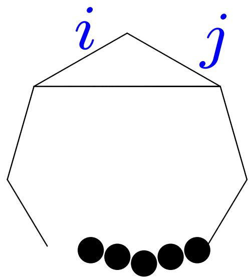
Standard orientation of $\mathbb{R}^{n-3}$ $\Rightarrow$ orientation on cells of MAX dim

Cells of the maximal dimension



, where $1 \le i \le n$, $4 \le j \le n$.

Boundaries by glue i, j, $1 \le i \le n$, $4 \le j \le n$, cells $K_{ij|l_1 \ldots l_{n-2}}$:



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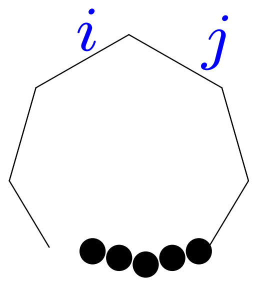

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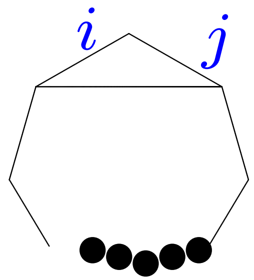
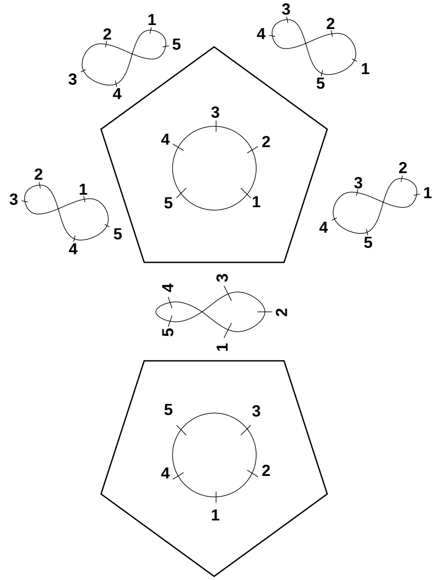
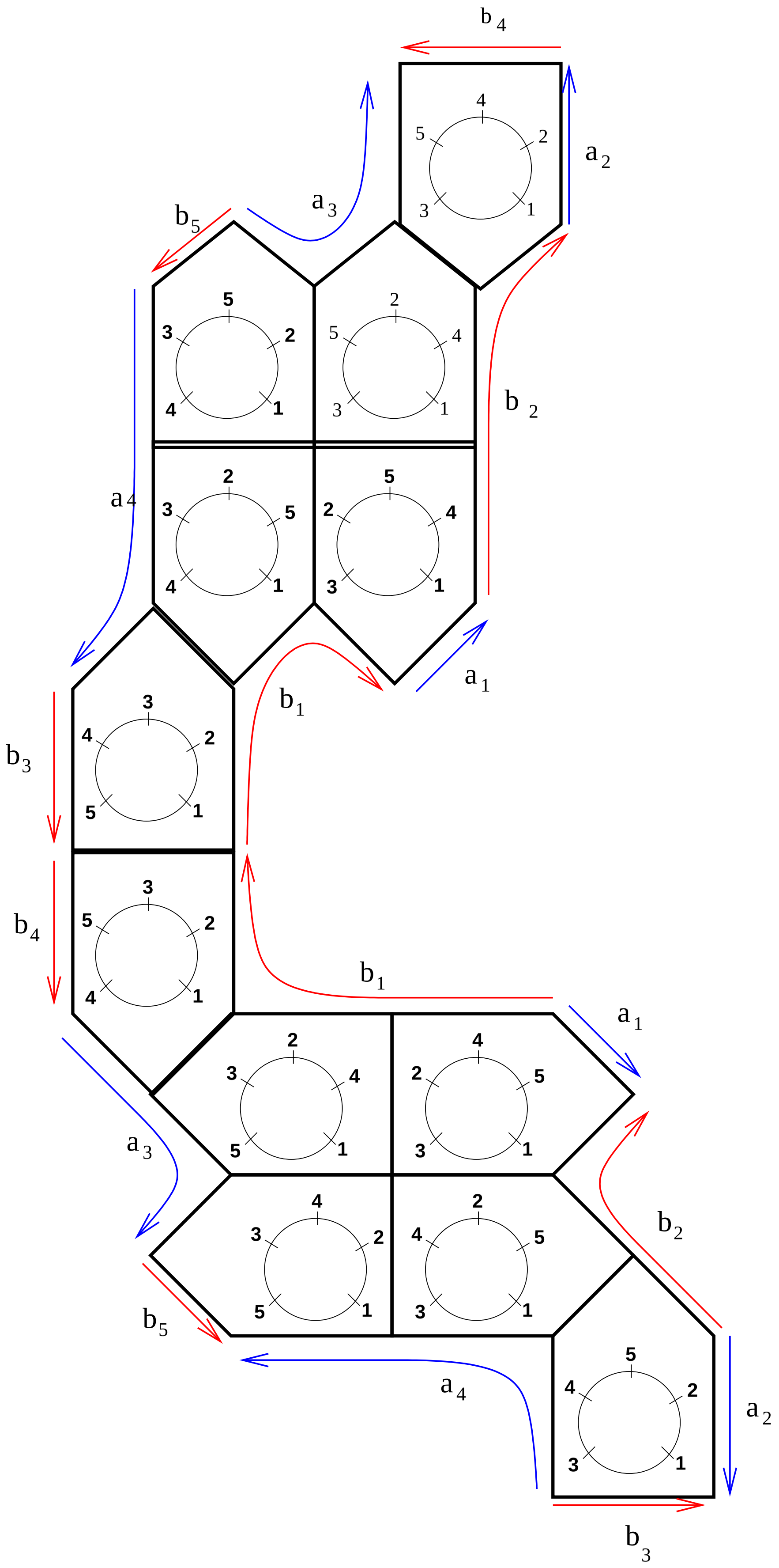
Boundaries by glue i, j, $1 \leq i \leq n$, $4 \leq j \leq n$, cells $K_{ij|l_1 \ldots l_{n-2}}$:



, where $1 \leq i \leq n$, $4 \leq j \leq n$.

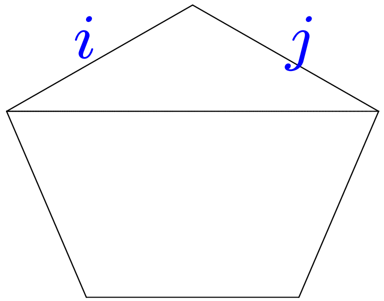
Lemma. $\forall\ n \geq 5$ and $\forall\ i, j$, $1 \leq i \leq n$, $4 \leq j \leq n$ the cells $K_{ij|l_1 \ldots l_{n-2}}$ are not in the class $W_{n-4}(\overline{\mathcal{M}^{\mathbb{R}}_{0,n}})$.

Proof. $\forall\ i,j,\ 1 \le i \le n,\ 4 \le j \le n$ the cells of MAX dim with common boundary $K_{ij|l_1 \dots l_{n-2}}$ look like  So, $K_{ij|l_1 \dots l_{n-2}}$ is in the sum twice with the opposite signs, hence, it is not in the class $W_{n-4}(\overline{\mathcal{M}^{\mathbb{R}}_{0,n}})$. $\square$



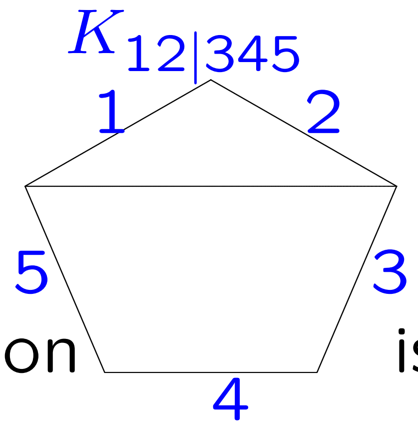
$n = 5$

By Lemma it remains to consider the cells of codim 1 of the form



where $1 \leq i, j \leq 3$.

We start with $i = 1$, $j = 2$.

Lemma. The boundary cell labeled by the pentagon 

is in the class $W_1(\overline{\mathcal{M}_{0,5}^{\mathbb{R}}})$.

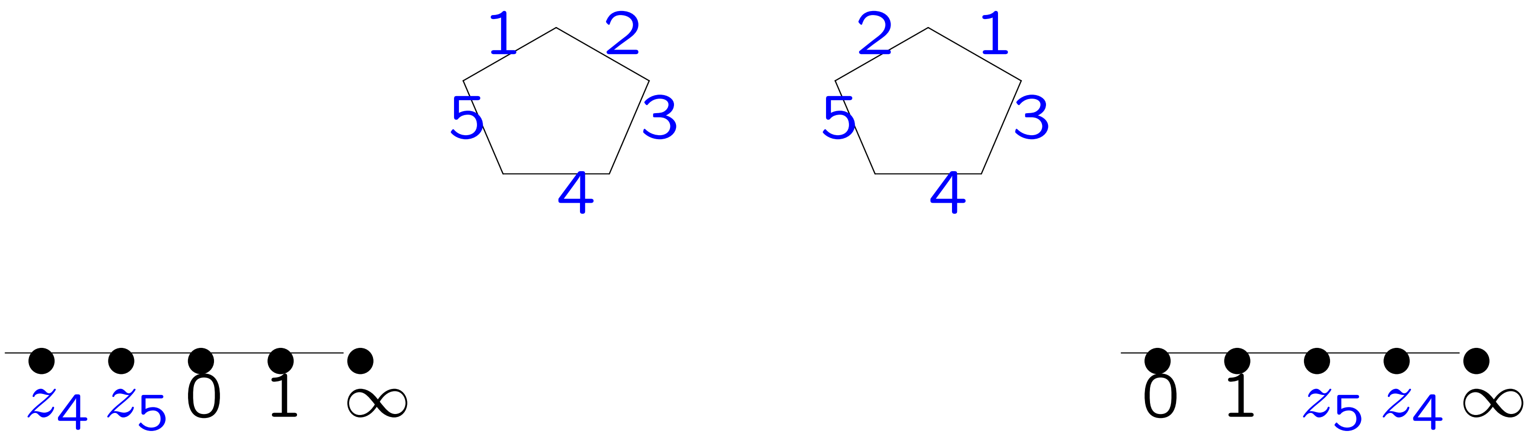
Proof. Consider the coordinates on $\mathcal{M}_{0,5}^{\mathbb{R}}$ which can be prolonged to this boundary: $\phi_1 : \mathcal{M}_{0,5}^{\mathbb{R}} \to \mathbb{R}^2$, $(\mathbb{P}_1(\mathbb{R}), y_1, \ldots, y_5) \in \mathcal{M}_{0,5}^{\mathbb{R}}$ with parametrization $\mathbb{P}_1(\mathbb{R})$: $y_3 = \infty, y_4 = 0, y_5 = 1$. We set $\phi_1(\mathbb{P}_1(\mathbb{R}), y_1, \ldots, y_5) = (y_1, y_2)$.

i	1	2	3	4	5
$z-$coordinates	0	1	∞	z_4	z_5
$y-$coordinates	y_1	y_2	∞	0	1

We seek $f(t) = \frac{at+b}{ct+d}$ and compute the Jacobian:

$$J = \det \begin{pmatrix} \frac{-z_5}{(z_4-z_5)^2} & \frac{z_4}{(z_4-z_5)^2} \\ \frac{1-z_5}{(z_4-z_5)^2} & \frac{1+z_4}{(z_4-z_5)^2} \end{pmatrix} = \frac{1}{(z_4 - z_5)^4}(z_5 - z_4).$$

$K_{12|345}$ is the common boundary of the following two cells:



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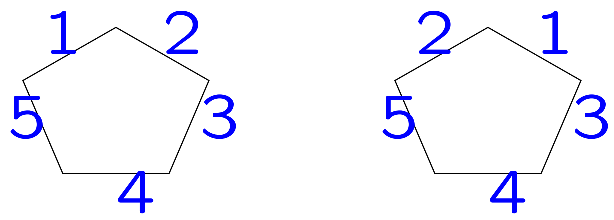
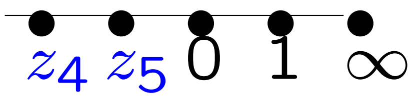
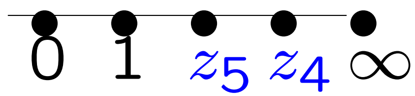


$J > 0$





$J < 0$

In the parametrization (y_1, y_2): opposite orientations,

Jacobians have the opposite signs $\Rightarrow$

In the parametrization (z_4, z_5): the same orientation.

Hence, $K_{12|345}$ is included twice to the expression for $W_1(\overline{\mathcal{M}^{\mathbb{R}}_{0,5}})$.

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$$W_{d-1}(M) = \left(\frac{1}{2}\sum \partial\overline{k_j}\right) \mod 2.$$

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$$W_{d-1}(M) = \left(\frac{1}{2}\sum \partial\overline{k_j}\right) \mod 2.$$

$\Rightarrow$ $W_1(\overline{\mathcal{M}^{\mathbb{R}}_{0,5}})$ contains the cell labeled by $K_{12|345}$. $\square$

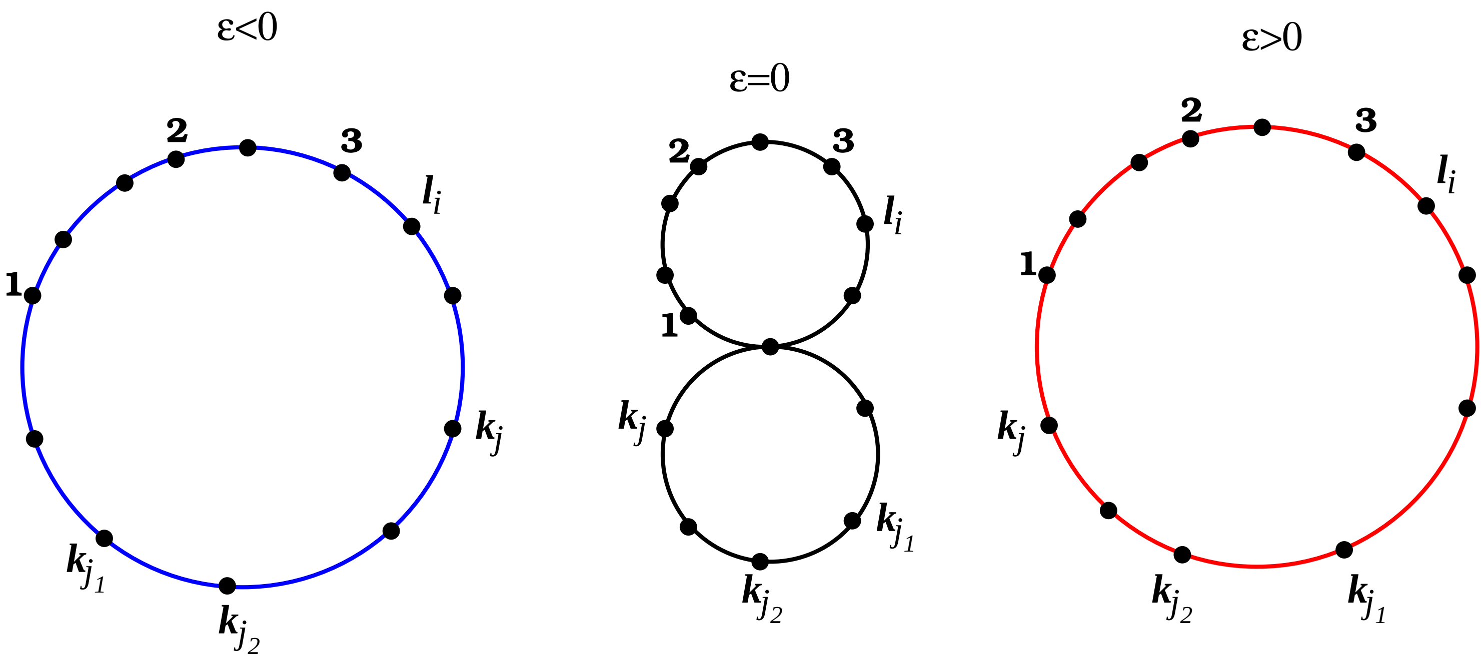
Computation of $W_{n-4}(\overline{\mathcal{M}^{\mathbb{R}}_{0,n}})$ for $n \geq 6$

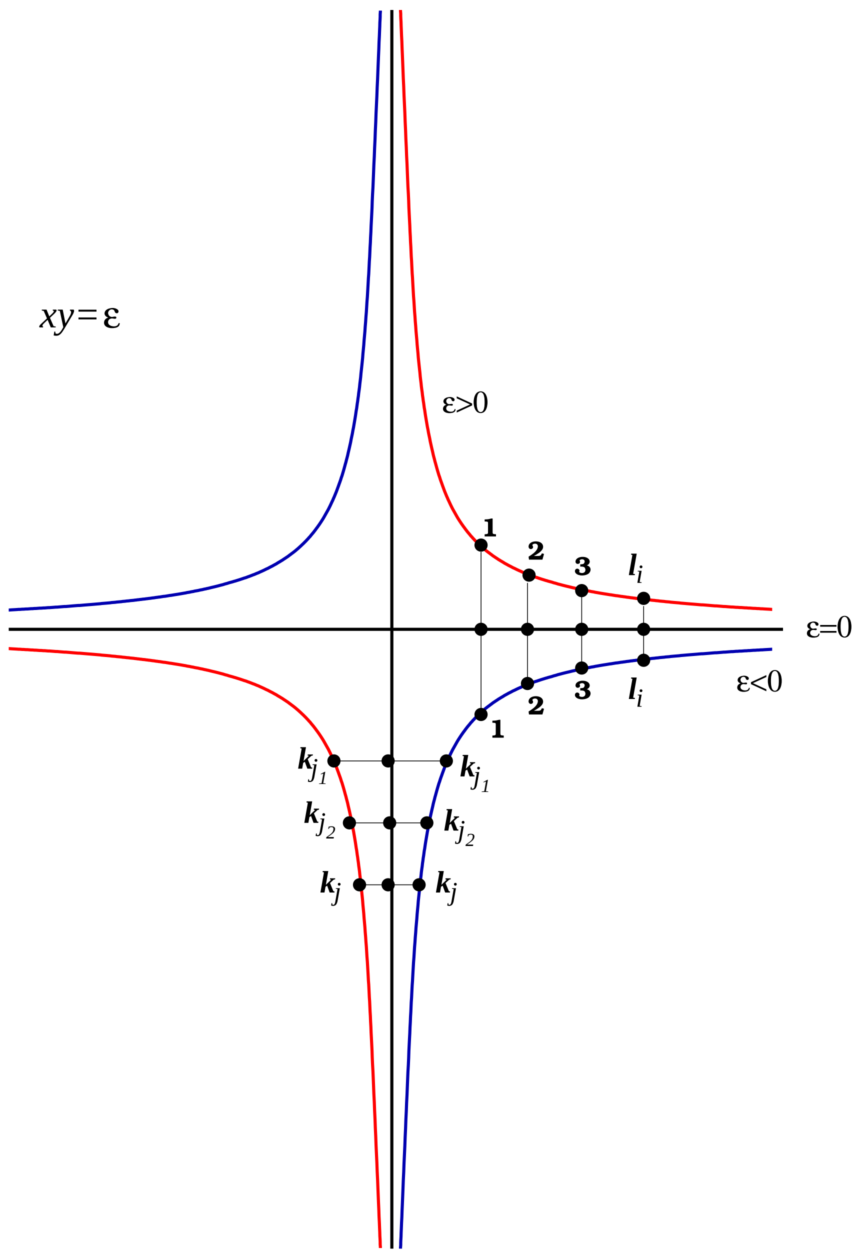
Special coordinates!

Draw the curve $\mathbb{P}_1(\mathbb{R})$ in the form of hyperbola $xy = \varepsilon$.

Approaching the boundary — taking the limit $\varepsilon \to 0$ under the fixed x or y of marked points.

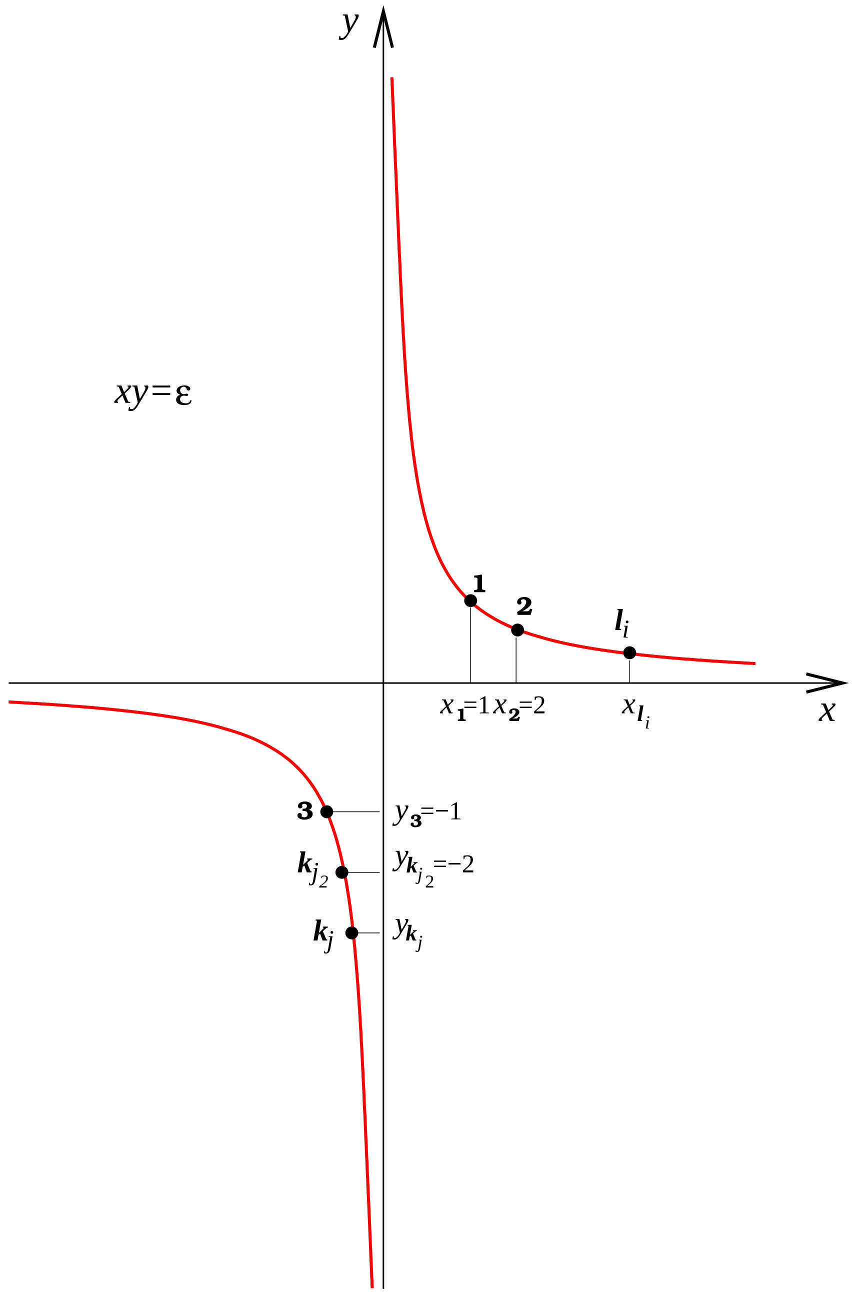
$xy = \varepsilon$





xy=ε
ε>0
ε=0
ε<0
1
2
3
l_i
1
2
3
l_i
k_{j_1}
k_{j_1}
k_{j_2}
k_{j_2}
k_j
k_j

Lemma. Let $n \geq 6$ and all three points 1, 2, 3 are in one component of the boundary. Then this cell is in the class $W_{n-4}(\overline{\mathcal{M}_{0,n}^{\mathbb{R}}})$ iff the boundary component does not containing 1, 2, 3, contain odd number of the marked points.



y
x
xy=ε
1
2
l_i
x_1=1
x_2=2
x_{l_i}
3
y_3=−1
k_{j_2}
$y_{k_{j_2}}$=−2
k_j
y_{k_j}

Lemma. Let $n \geq 6$ and only two of the points 1, 2, 3 are in the same component of the boundary K. Then K is in $W_{n-4}(\overline{\mathcal{M}^{\mathbb{R}}_{0,n}})$ iff $\exists$ odd number of marked points on the component of K which contains the third point from the set $\{1, 2, 3\}$.